A REMARK ON AN AREALLY MEAN p-VALENT FUNCTION

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ABSTRACT

We bring an example which shows that in a theorem due to Cartwright, Spencer and Hayman concerning areally mean p-valent functions a multiplicative constant cannot be reduced to 1. (This is possible in the corresponding theorem for circumferentially mean p-valent functions).

We first recall the following definition [1]. Suppose that

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

is regular and not zero in $|z| < 1$. Denote by D the domain on which $|z| < 1$ is mapped by the function $f(z)$. Denote by $W(R)$ the area (counted with the appropriate multiplicity) of that part of D which is enclosed in the circle $|w| \leqq R$.

A function $f(z)$ for which $W(R) \leqq p \cdot \pi R^2$ for every positive R is said to be areally mean p valent. (a.m.p.v.). We also recall the following theorem: Suppose that $f(z)$ is an a.m.p.v function and not zero in $|z| < 1$ Then we have [1]:

$$\frac{|a_0|}{c}\left(\frac{1-|z|}{1+|z|}\right)^{2p} < |f(z)| < |a_0| c \left(\frac{1+|z|}{1-|z|}\right)^{2p}, \quad 0 < |z| < 1, \tag{1}$$

where $c = e^{2\pi p + 1/2}$

Our aim is now to show the following

THEOREM 1. *Suppose that for every function $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ a.m.p.v. and not zero in $|z| < 1$, we have the following relation:*

$$\frac{|a_0|}{d_2 e^{m_2 p}} \cdot \left(\frac{1-|z|}{1+|z|}\right)^{2p} < |f(z)| < |a_0| d_1 e^{m_1 p}\left(\frac{1+|z|}{1-|z|}\right)^{2p}, \quad 0 < |z| < 1, \tag{2}$$

then it follows that $d_1, d_2 \geqq 2/\sqrt{e}$.

Proof. The inequality $d_1 \geqq 2/\sqrt{e}$ will be shown with the aid of the function

$$f(z) = k + \log\left(1 + \frac{1+z}{1-z}\right), \; k > 0.$$

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We have: $\operatorname{Re} f(z) \geqq k$, $|\operatorname{Im} f(z)| < \pi/2$.

Clearly $f(z)$ is univalent. Thus the area of the part of the image of $|z| < 1$ by $f(z)$ which lies over $|w| < R$, does not exceed the area of the rectangle $|v| < \pi/2$, $k < u < R$, i.e.$\pi(R-k)$. Since

$$\frac{\pi(R-k)}{\pi R^2} \leqq \frac{1}{4k} \qquad k \leqq R < \infty,$$

$f(z)$ is mean p-valent, with $p = 1/4k$. Thus we have from (2):

$$\frac{\left|k + \log\left(1 + \frac{1+z}{1-z}\right)\right|}{k + \log 2} < d_1 e^{m_1/4k}\left(\frac{1+|z|}{1-|z|}\right)^{1/2k}. \tag{3}$$

We now choose z such that $0 < z < 1$ and $(1 + z/1 - z) = e^k$. Then from (3) we have:

$$\frac{k + \log(1 + e^k)}{k + \log 2} < d_1 e^{m_1/4k} e^{1/2}. \tag{4}$$

If now $k \to \infty$, we get from (4): $2 \leqq d_1 e^{1/2}$.

For the proof of the second inequality we define:

$$f(z) = k + \log\left(l + \frac{1+z}{1-z}\right), k, l > 0,\ k + \log l > 0.$$

By similar considerations to the first case, the function $f(z)$ is a.m.p.v. for $p = 1/(4(k + \log l))$ and we may use (2) for this value of p. So:

$$\frac{\left|k + \log\left(l + \frac{1+z}{1-z}\right)\right|}{k + \log(l+1)} > \frac{1}{d_2 e^{m_2/4t}}\left(\frac{1-|z|}{1+|z|}\right)^{1/2t} \quad \text{where } t = k + \log l.$$

We now take $-1 < z < 0$, and set $u = (1 + |z|/1 - |z|)$. Then:

$$d_2 > \frac{t + \log(1 + 1/l)}{e^{m_2/4t} u^{1/2t}[t + \log(1 + 1/ul)]} \tag{5}$$

If, in particular, $1/l = u = e^t$, we get:

$$d_2 > \frac{t + \log(1 + e^t)}{e^{m_2/4t} e^{1/2}(t + \log 2)}. \tag{6}$$

If now $t \to \infty$, we have $d_2 \geqq 2/\sqrt{e}$, and the proof is complete.

Reference

1. Hayman, W. K. *Multivalent functions* 1st ed., Cambridge University Press (1958).

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